Chapter 14

Isotherman Parameters

Let $x : U \to S$ be a regular surface. Let

$$\phi_k(z) = \frac{\partial x_k}{\partial u_1} - i\frac{\partial x_k}{\partial u_2}, z = u_1 + iu_2. \tag{14.1}$$

Recall from last lecture that

a)ϕ is analytic in z $\Leftrightarrow$ x_k is harmonic in u_1 and u_2.

b)u_1 and u_2 are isothermal parameters $\Leftrightarrow$

$$\sum_{k=1}^{n} \phi_k^2(z) = 0 \tag{14.2}$$

c) If u_1, u_2 are isothermal parameters, then S is regular $\Leftrightarrow$

$$\sum_{k=1}^{n} |\phi_k(z)|^2 \neq 0 \tag{14.3}$$

We start by stating a lemma that summarizes what we did in the last lecture:

Lemma 4.3 in Osserman: Let x(u) define a minimal surface, with u_1, u_2 isothermal parameters. Then the functions $\phi_k(z)$ are analytic and they satisfy the eqns in b) and c). Conversely if $\phi_1, \phi_2, .., \phi_n$ are analytic functions satisfying the eqns in b) and c) in a simply connected domain D

then there exists a regular minimal surface defined over domain D, such that the eqn on the top of the page is valid.

Now we take a surface in non-parametric form:

$$x_k = f_k(x_1, x_2), k = 3, ..., n \tag{14.4}$$

and we have the notation from the last time:

$$f = (f_3, f_4, ..., f_n), p = \frac{\partial f}{\partial x_1}, q = \frac{\partial f}{\partial x_2}, r = \frac{\partial^2 f}{\partial x_1^2}, s = \frac{\partial^2 f}{\partial x_1 \partial x_2}, t = \frac{\partial^2 f}{\partial x_2^2} \tag{14.5}$$

Then the minimal surface eqn may be written as:

$$(1 + |q|^2)\frac{\partial p}{\partial x_1} - (p.q)(\frac{\partial p}{\partial x_2} + \frac{\partial q}{\partial x_1}) + (1 + |p|^2)\frac{\partial q}{\partial x_2} = 0 \tag{14.6}$$

equivalently

$$(1 + |q|^2)r - 2(p.q)s + (1 + |p|^2)t = 0 \tag{14.7}$$

One also has the following:

$$det g_{ij} = 1 + |p|^2 + |q|^2 + |p|^2|q|^2 - (p.q)^2 \tag{14.8}$$

Define

$$W = \sqrt{det g_{ij}} \tag{14.9}$$

Below we'll do exactly the same things with what we did when we showed that the mean curvature equals 0 if the surface is minimizer for some curve. Now we make a variation in our surface just like the one that we did before (the only difference is that x_1 and x_2 are not varied.)

$$\tilde{f}_k = f_k + \lambda h_k, k = 3, ..., n, \tag{14.10}$$

where λ is a real number, and $h_k \in C^1$ in the domain of definition D of the

f_k We have

$$\tilde{f} = f + \lambda h, \tilde{p} = p + \lambda \frac{\partial h}{\partial x_1}, \tilde{q} = q + \lambda \frac{\partial h}{\partial x_2} \tag{14.11}$$

One has

$$\tilde{W}^2 = W^2 + 2\lambda X + \lambda^2 Y \tag{14.12}$$

where

$$X = [(1 + |q|^2)p - (p.q)q]. \frac{\partial h}{\partial x_1} + [(1 + |p|^2)q - (p.q)p]. \frac{\partial h}{\partial x_2} \tag{14.13}$$

and Y is a continuous function in x_1 and x_2. It follows that

$$\tilde{W} = W + \lambda \frac{X}{W} + O(\lambda^2) \tag{14.14}$$

as $|\lambda| \to 0$ Now we consider a closed curve Γ on our surface. Let Δ be the region bounded by Γ If our surface is a minimizer for Δ then for every choice of h such that $h = 0$ on Γ we have

$$\int \int_\Delta \tilde{W} dx_1 dx_2 \geq \int \int_\Delta W dx_1 dx_2 \tag{14.15}$$

which implies

$$\int \int_\Delta \frac{X}{W} = 0 \tag{14.16}$$

Substituting for X, integrating by parts, and using the fact that $h = 0$ on Γ , we find

$$\int \int_\Delta \left[\frac{\partial}{\partial x_1} \left[\frac{1 + |q|^2}{W} p - \frac{p.q}{W} q \right] + \frac{\partial}{\partial x_2} \left[\frac{1 + |p|^2}{W} q - \frac{p.q}{W} p \right] \right] h dx_1 dx_2 = 0 \tag{14.17}$$

must hold everywhere. By the same reasoning that we used when we found the condition for a minimal surface the above integrand should be zero.

$$\frac{\partial}{\partial x_1} \left[\frac{1 + |q|^2}{W} p - \frac{p.q}{W} q \right] + \frac{\partial}{\partial x_2} \left[\frac{1 + |p|^2}{W} q - \frac{p.q}{W} p \right] = 0 \tag{14.18}$$

Once we found this equation it makes sense to look for ways to derive it from the original equation since after all there should only be one constraint for a minimal surface. In fact the LHS of the above eqn can be written as the sum of three terms:

$$\left[\frac{1+|q|^2}{W}\frac{\partial p}{\partial x_1}-\frac{p.q}{W}(\frac{\partial q}{\partial x_1}+\frac{\partial p}{\partial x_2})+\frac{1+|p|^2}{W}\frac{\partial q}{\partial x_2}\right] \tag{14.19}$$

$$+\left[\frac{\partial}{\partial x_1}(\frac{1+|q|^2}{W})-\frac{\partial}{\partial x_2}(\frac{p.q}{W})\right]p \tag{14.20}$$

$$+\left[\frac{\partial}{\partial x_2}(\frac{1+|p|^2}{W})-\frac{\partial}{\partial x_1}(\frac{p.q}{W})\right]q \tag{14.21}$$

The first term is the minimal surface eqn given on the top of the second page. If we expand out the coefficient of p in the second term we find the expression:

$$\frac{1}{W^3}[(p.q)q-(1+|q|^2)p].[(1+|q|^2)r-2(p.q)s+(1+|p|^2)t] \tag{14.22}$$

which vanishes by the second version of the minimal surface eqns. Similarly the coefficient of q in third term vanishes so the while expression equals zero. In the process we've also shown that

$$\frac{\partial}{\partial x_1}\left(\frac{1+|q|^2}{W}\right)=\frac{\partial}{\partial x_2}\left(\frac{p.q}{W}\right) \tag{14.23}$$

$$\frac{\partial}{\partial x_2}\left(\frac{1+|p|^2}{W}\right)=\frac{\partial}{\partial x_1}\left(\frac{p.q}{W}\right) \tag{14.24}$$

Existence of isothermal parameters or Lemma 4.4 in Osserman Let S be a minimal surface. Every regular point of S has a neighborhood in which there exists a reparametrization of S in terms of isothermal parameters.

Proof: Since the surface is regular for any point there exists a neighborhood of that point in which S may be represented in non-parametric form. In particular we can find a disk around that point where the surface can be

represented in non parametric form. Now the above eqns imply the existence of functions $F(x_1, x_2)$ $G(x_1, x_2)$ defined on this disk, satisfying

$$\frac{\partial F}{\partial x_1} = \frac{1+|p|^2}{W}, \frac{\partial F}{\partial x_2} = \frac{p.q}{W}; \tag{14.25}$$

$$\frac{\partial G}{\partial x_1} = \frac{p.q}{W}, \frac{\partial G}{\partial x_2} = \frac{1+|q|^2}{W} \tag{14.26}$$

If we set

$$\xi_1 = x_1 + F(x_1, x_2), \xi_2 = x_2 + G(x_1, x_2), \tag{14.27}$$

we find

$$J = \frac{\partial(x_1, x_2)}{\partial(x_1, x_2)} = 2 + \frac{2+|p|^2+|q|^2}{W} \geq 0 \tag{14.28}$$

Thus the transformation $(x_1, x_2) \to (\xi_1, \xi_2)$ has a local inverse $(\xi_1, \xi_2) \to (x_1, x_2)$. We find the derivative of x at point (ξ_1, ξ_2):

$$Dx = J^{-1}[x_1, x_2, f_3, ..., f_n] \tag{14.29}$$

It follows that with respect to the parameters ξ_1, ξ_2 we have

$$g_{11} = g_{22} = |\frac{\partial x}{\partial \xi_1}|^2 = |\frac{\partial x}{\partial \xi_2}|^2 \tag{14.30}$$

$$g_{12} = \frac{\partial x}{\partial \xi_1}.\frac{\partial x}{\partial \xi_2} = 0 \tag{14.31}$$

so that ξ_1, ξ_2 are isothermal coordinates.